

# INCREASING THE NUMBER AND VOLUME OF HEXAHEDRAL AND PRISM ELEMENTS IN A HEX-DOMINANT MESH BY TOPOLOGICAL TRANSFORMATIONS

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## ABSTRACT

This paper describes a new method for increasing the number and the volume of hexahedral and prism elements in a hex-dominant mesh by topological transformations. The method takes as input a hex-dominant mesh consisting of hexahedrons, prisms, pyramids and tetrahedrons and modifies the mesh to increase the number and the volume of hexahedrons and prisms while maintaining the relaxed conformity criteria, which allows a connection from two tetrahedrons to a quadrilateral face of a hexahedron or a prism. If a hex-dominant mesh satisfies the relaxed conformity criteria, it can be used in the finite element analysis by applying an error reduction scheme on non-conforming faces [1-3], inserting pyramids on non-conforming faces [4], or converting the mesh to an all-hex mesh by a template method [5, 6]. With more hexahedrons and prisms in a hex-dominant mesh, a more accurate finite element solution can be obtained in a shorter time. Hence the proposed method increases the practical value of a hex-dominant mesh. Several experiments showed the number of hexahedrons increased by about 10% to 20%, yielding hex-dominant meshes with 70% to 90% hexahedron volume ratio.

**Keywords:** hex-dominant mesh, topological transformation

## 1. INTRODUCTION

This paper describes a new method for increasing the number and the volume of hexahedral elements in a hex-dominant mesh by transforming elements. In general, a hex-dominant mesh yields a more accurate finite element solution with a shorter computational time than a tetrahedral mesh with the same degrees of freedom, and is easier to create automatically than an all-hex mesh. However, even with an equal degrees of freedom, the accuracy and computational time of the finite element analysis depends on the ratio of the number and the volume of hexahedrons included in the mesh; increasing the number of hexahedrons increases the accuracy of the solution and decreases the computational time. Hence, it is important to create a hex-dominant mesh with as many hexahedrons as possible and with as much volume as possible being hexahedrons. Or, if possible, a post-

process for a hex-dominant mesh would be useful to increase the number and volume of hexahedrons. (Please note that applying subdivision patterns used in [7-9] increases the number of hexahedrons, but it does not increase the total volume filled by hexahedrons.)

The proposed method takes as input a hex-dominant mesh consisting of hexahedrons, prisms, pyramids and tetrahedrons, and increases the number and the volume of hexahedrons and prisms while maintaining *the relaxed conformity criteria*, which allows a connection from two tetrahedrons to a quadrilateral face of a hexahedron or a prism. The details of the relaxed conformity criteria are discussed in Section 3. The method first subdivides prisms and pyramids included in the input mesh into tetrahedrons. Then, the method applies sequences of topological transformations (explained in Section 4) to increase the number of hexahedrons. Some of the remaining tetrahedrons are then merged and converted to prisms [10]. If the mesh must conform, pyramid elements can be

inserted between tetrahedrons and a quadrilateral face of a hexahedron or a prism via the method presented by Owen et al. [4].

The organization of the paper is as follows. Section 2 reviews previous work of hex-dominant mesh generation. Section 3 discusses the relaxed conformity criteria and gives a definition of a hex-dominant mesh which is dealt with in the proposed method. Section 4 describes three types of topological transformations that modifies some elements while maintaining the relaxed conformity criteria. Section 5 presents a strategy for applying the transformations to increase the number and volume of hexahedrons. Section 6 gives some results of the method, followed by discussions in Section 7 and conclusions in Section 8.

## 2. RELATED RESEARCH

A few hex-dominant mesh generation techniques have been published. Meyers et al. [11] and Tuchinsky and Clark [12] present a method that creates a hex-dominant mesh by expanding the *plastering method*, which is presented by Blacker and Meyers [13]. Their method creates hexahedrons from the boundary inward by the plastering method, and if the plastering method cannot fill entire volume with hexahedrons, their method fills the remaining volume with tetrahedrons.

Owen and Saigal present an algorithm called *H-Morph* [14], which converts a tetrahedral mesh to a hex-dominant mesh by creating hex elements one by one starting from domain boundaries and moving inward. The method always maintains a valid hexahedron-tetrahedron mixed mesh during the process.

Meshkat and Talmor present an algorithm that converts a tetrahedral mesh into a hex-dominant mesh based on the graph theory [15]. Their method takes a tetrahedral mesh as input and creates a graph that represents the topology of the tetrahedral mesh. Their method then searches for a pattern that can be converted to a hex or a prism in the graph, and when a pattern is found, a new hex or a new prism is created, and the graph is updated accordingly.

Yamakawa and Shimada present a method that first creates good node locations for a hex-dominant mesh by packing rectangular solid cells in the target geometric domain. A tetrahedral mesh is then created using the node obtained by the packing. Finally, the tetrahedral mesh is converted to a hex-dominant mesh by merging some tetrahedrons [10].

There is a highly effective post-process for a tetrahedral mesh called *local transformation* [16], which improves the quality of a tetrahedral mesh, though, no such method has been published for a hex-dominant mesh. Bern and Eppstein present one method of transforming hexahedral elements [17]. However, it is unclear if their method can improve the quality of an all-hex mesh or a hex-dominant mesh.

## 3. RELAXED CONFORMITY CRITERIA

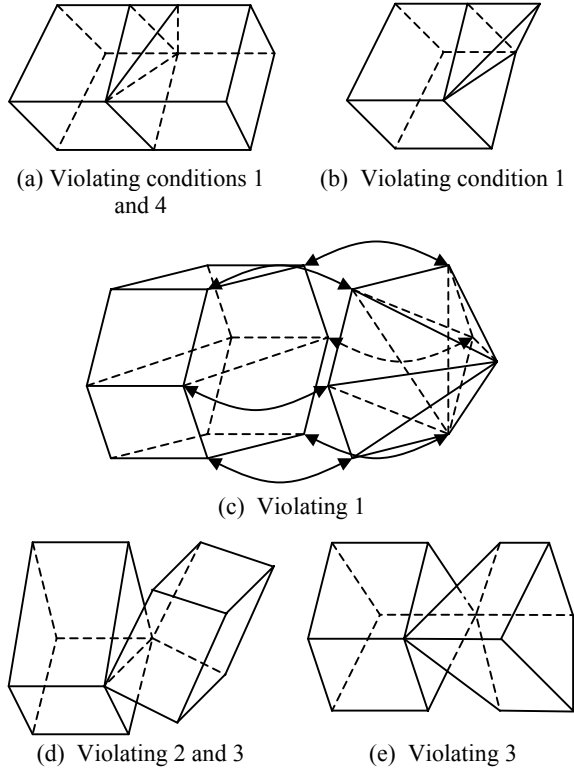
The topological transformations described in Section 4 deal with a hex-dominant mesh that satisfies the *relaxed conformity criteria*. The criteria allows a connection between two tetrahedrons and a hexahedron or a prism through a quadrilateral face. The purpose of the relaxed conformity criteria is to increase the applicability of the mesh to the finite element analysis and relieve the burden of hex-dominant mesh generation. The traditional conformity criterion demands that an interface between two elements be identical and that the interface must not be shared by a third element. A tetrahedron thus cannot be directly connected to a hexahedron by a face because a hexahedron has only quadrilateral faces, and a tetrahedron has only triangular faces; this condition makes a hex-dominant mesh very difficult to create. To relieve the burden of hex-dominant mesh generation, the relaxed conformity criteria is introduced, and it allows connection between two triangular faces of two tetrahedrons to a quadrilateral face of a hexahedron by sharing the four nodes. Such a quadrilateral face connected to two triangular faces is called a *non-conforming face*, or more explicitly, a *non-conforming quadrilateral*.

Although the relaxed conformity criteria has been implicitly used [1-3, 12, 15], a clear definition of the relaxed conformity criteria has not been given; here we propose a definition of the relaxed conformity conditions.

- Condition 1. For a non-conforming quadrilateral, there must be exactly two tetrahedrons that each has only one triangle sharing three of the four nodes of the non-conforming quadrilateral. The quadrilateral obtained by merging the two triangles must be equal to the non-conforming quadrilateral.
- Condition 2. A diagonal of a non-conforming quadrilateral must not be an edge of another quadrilateral.
- Condition 3. If two quadrilaterals share only two nodes, the two quadrilaterals must share an edge (i.e., sharing only a diagonal is not permissible.)
- Condition 4. Two quadrilaterals must not share only three nodes.

For example, the case illustrated in Figure 1 (a) violates Conditions 1 and 4 because the left hexahedron is connected to the right hexahedron and the tetrahedron between the two hexahedrons through a quadrilateral face, and the quadrilateral of the right face of the left hexahedron and a quadrilateral of the left face of the right hexahedron shares only three nodes. The case illustrated in Figure 1 (b) violates Condition 1 because only one tetrahedron is connected to the right-hand side quadrilateral of the hexahedron, and half of the quadrilateral is exposed to the exterior of the mesh. The case illustrated in Figure 1 (c) violates Condition 1 because the right-hand side quadrilateral of the top hexahedron is connected to three tetrahedrons, and the right-hand side quadrilateral of the bottom hexahedron is connected to four tetrahedrons. The case illustrated in Figure 1 (d) violates Conditions 2 and 3

because an edge of a quadrilateral is lying on the diagonal of another quadrilateral. The case illustrated in Figure 1 (e) violates Condition 3 because the right-hand side quadrilateral of the left hexahedron and the left-hand side quadrilateral of the right hexahedron share two nodes, but do not share an edge.



**Figure 1 Examples of violation of the relaxed conformity conditions**

If a hex-dominant mesh satisfies the four relaxed conformity conditions, it can be used in the finite element analysis by:

- Method 1. Inserting a pyramid on each non-conforming quadrilaterals to recover perfect conformity via the method presented by Owen et al. [4],
- Method 2. Applying an error reduction scheme, such as MPCs [1, 2] or Dohrmann et al.'s [3], or
- Method 3. Converting it to an all-hex mesh by applying conversion templates, such as HEXHOOP templates [5] or Geode template [6]. (Geode template requires another condition in addition to the relaxed conformity criteria to be applicable; all non-conforming quadrilaterals must form a topological ball or terminate at the boundary. Nonetheless, satisfying the relaxed conformity criteria makes Geode template more likely to be applicable.)

Applicability of the above methods also depends on a finite element solver. Method 1 can be applied when a solver can take pyramid elements. Method 2 can be applied when a

solver is capable of imposing an error reduction scheme. Method 3 can be applied when a solver is not too restrictive about the quality of the all-hex mesh converted from a hex-dominant mesh. Nonetheless, satisfying the relaxed conformity criteria makes a hex-dominant mesh more likely to be applicable to the finite element analysis.

The next section shows three types of topological transformations for a hex-dominant mesh that transform elements while maintaining the relaxed conformity criteria.

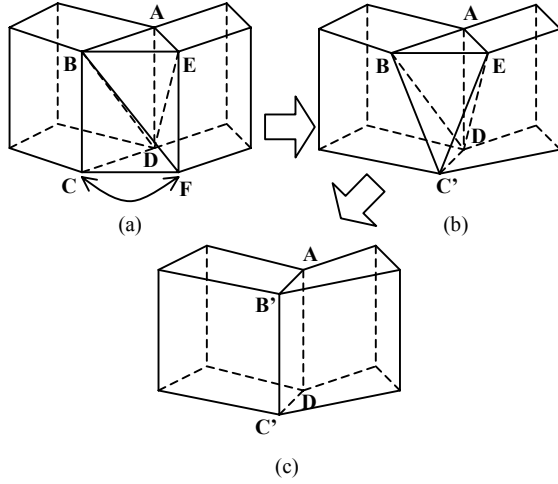
## 4. TOPOLOGICAL TRANSFORMATIONS FOR A HEX-DOMINANT MESH

This section explains three types of topological transformations developed in our research: (1) edge-collapse, (2) node insertion, and (3) shear. Each transformation is performed while maintaining the relaxed conformity criteria defined in Section 3. Applying these transformations systematically increases the number and volume of hexahedrons in a hex-dominant mesh. The strategy of applying these transformations is explained later in Section 5.

### 4.1. Edge-collapse transformation

The edge-collapse transformation collapses an edge that is used only by tetrahedrons. All tetrahedrons using the edge are deleted and the two nodes of the edge are joined into one node. If the two nodes of the edge are not used by any hexahedrons, the transformation can be easily accomplished by merging two nodes into one and deleting all degenerating tetrahedrons. However, if one of the nodes of the edge is used by a hexahedron, the relaxed conformity criteria may be violated as a result of the transformation, and thus the program must check for the violation.

When the relaxed conformity criteria are violated as a result of the edge-collapse transformation, the violation may be resolved by applying a subsequent edge-collapse transformation. Figure 2 shows an example of such resolution. Figure 2 (a) shows a portion of a mesh, two hexahedrons and three tetrahedrons between them. When edge **CF** is collapsed, the relaxed conformity criteria are violated because quadrilaterals **ABC'D** and **ADC'E** shares only three nodes as shown in Figure 2 (b). To resolve the violation, edge **BE** must be collapsed as shown in Figure 2 (c). The second edge-collapse transformation may yield another violation of the relaxed conformity criteria, and if so, the method attempts to resolve the violation by applying an additional edge-collapse transformation. Thus, one edge-collapse transformation may yield a series of the edge-collapse transformations, which continue until all violations are resolved. However, the violation is not always resolved by applying a series of the transformations, and if a violation cannot be resolved in the end, the series of the transformations must be cancelled.



**Figure 2 Edge-collapse transformation: Resolving a violation of the relaxed conformity criteria by a series of edge-collapse transformations.**

#### 4.2. Node insertion

The node insertion transformation adds a node on an edge and subdivides elements using the edge. If a target edge (an edge on which a node is inserted) is used only by tetrahedrons, each tetrahedron using the edge is simply subdivided into two tetrahedrons by a triangle formed by the node inserted on the edge and the two nodes of the original tetrahedron that are not connected to the edge.

Inserting a node on an edge used by a hexahedron, however, is not easily accomplished because four subdivision patterns are possible; the appropriate pattern must be chosen for a configuration around the hexahedron, or the transformation cannot be applied if none of the patterns fits the configuration. As shown in Figure 3, when node  $X$  is inserted on edge  $FG$  of hexahedron  $ABCDEFGH$ , the hexahedron can split into four possible subdivision patterns:

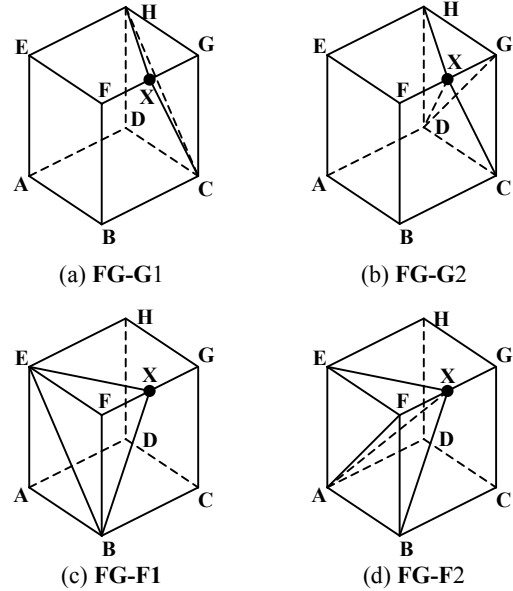
- (1) one hexahedron  $ABCDEFXH$  and one tetrahedron  $CHGX$  as shown in Figure 3 (a),
- (2) one hexahedron  $ABCDEFXH$  and two tetrahedrons  $DHGX$  and  $CDGX$  as shown in Figure 3 (b),
- (3) one hexahedron  $ABCDEXGH$  and one tetrahedron  $BEXF$  as shown in Figure 3 (c), and
- (4) one hexahedron  $ABCDEXGH$  and two tetrahedrons  $AEXF$  and  $BAXF$  as shown in Figure 3 (d).

To distinguish these patterns, a subdivision pattern is denoted as  $PQ-Rn$ , which means a node is inserted on edge  $PQ$ , and  $R$  becomes a node of a new tetrahedron (where  $R$  is equal to either  $P$  or  $Q$ ), and  $n$  new tetrahedrons are created by the node insertion. Subdivision pattern (1) is thus denoted as  $FG-G1$ , pattern (2)  $FG-G2$ , pattern (3)  $FG-F1$  and pattern (4)  $FG-F2$ .

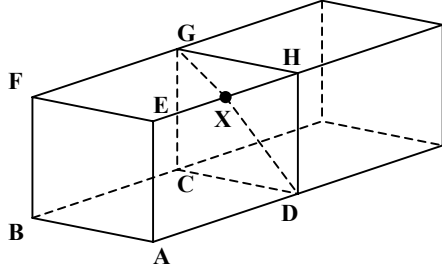
To apply pattern  $PQ-Rn$ , the face of the hexahedron using node  $R$  and not using edge  $PQ$  must be either a non-conforming quadrilateral or a quadrilateral exposed to the exterior of the mesh. For example, if pattern  $EH-H1$  or  $EH-H2$  is applied to the mesh shown in Figure 4, quadrilaterals  $DCGX$  and  $DCGH$  will share only three nodes after the node insertion, and condition (4) of the relaxed conformity criteria will be violated.

In addition, to apply pattern  $PQ-Rn$ , if a quadrilateral face of the hexahedron using edge  $PQ$  is a non-conforming quadrilateral, one of the edges of the two tetrahedrons connected to the quadrilateral face lying on a diagonal of the quadrilateral face must not use node  $R$ . For example, when two tetrahedrons  $ADBT$  and  $DCBT$  are connected to face  $ABCD$  as shown in Figure 5, applying pattern  $AB-Bn$  yields a violation of condition (1) of the relaxed conformity criteria because a non-conforming quadrilateral  $AXCD$  will not be connected to exactly two tetrahedrons by faces.

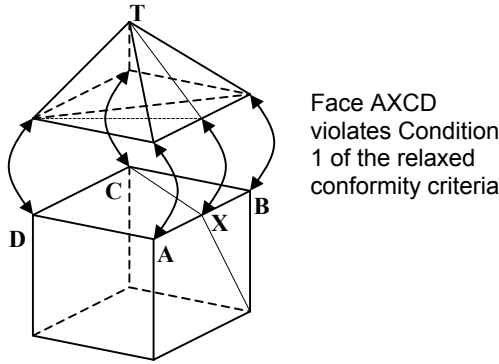
If all hexahedrons using edge  $PQ$  satisfy the above conditions for pattern  $PQ-Rn$  ( $n$  can vary across the hexahedrons), a node can be inserted on edge  $PQ$ . Note that yet the appropriate  $n$  must be chosen. If the quadrilateral face using node  $R$  and not using edge  $PQ$  is a non-conforming quadrilateral, one of the edges of the two tetrahedrons connected to the quadrilateral face must be lying on a diagonal of the quadrilateral face. If the edge of the tetrahedrons lying on a diagonal of the quadrilateral face is using node  $R$ ,  $n$  must be 2 as shown in Figure 6 (a). Otherwise,  $n$  must be 1 as shown in Figure 6 (b).



**Figure 3 Four possible subdivision patterns when inserting a node on an edge used by a hexahedron**



**Figure 4 Node insertion transformation: Applying EH-X1 or EH-X2 to this mesh yields violation of the relaxed conformity criteria**



**Figure 5 Node insertion transformation: The two tetrahedrons connected to face ABCD must not have edge BD**

If the quadrilateral face using node **R** and not using edge **PQ** is exposed to the exterior of the mesh,  $n$  must be 2, or a quadrilateral face of the new hexahedron will be connected to only one tetrahedron, creating a violation of condition (1) of the relaxed conformity criteria.

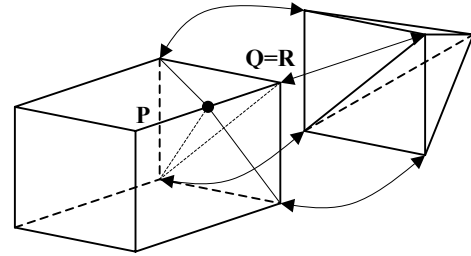
#### 4.3. Shear transformation

The shear transformation shears a hexahedron by re-connecting a hexahedron and tetrahedrons sharing an edge and/or a diagonal with one of the six faces of the hexahedron. Suppose some tetrahedrons sharing an edge and/or a diagonal of a face of a hexahedron can be merged and converted to a prism. For example, tetrahedrons **JFEI**, **IEDF** and **EADF** shown in Figure 7 (a) are sharing an edge and/or a diagonal of quadrilateral **IFAD** of hexahedron **ABCD EFGH** and can be merged and converted to a prism. Now the hexahedron can be split into two prisms by a quadrilateral formed by connecting two edges: (1) an edge connecting the two triangles of the pseudo-prism and being shared by the pseudo-prism and the hexahedron, and (2) the opposite edge of edge (1) in the hexahedron. In Figure 7 (a), edge (1) can be either **ID** or **FA**, and the quadrilateral can be **IDBG** or **FACH** respectively. One of the two prisms that came from a hexahedron shares a quadrilateral face with the pseudo-prism and can be merged with the pseudo-prism to become a hexahedron. The other prism

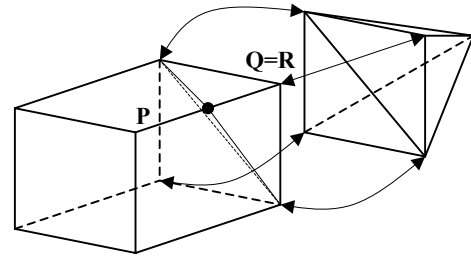
must be split into tetrahedrons while maintaining the relaxed conformity criteria. If quadrilateral **IDBG** is chosen to split the original hexahedron **ABCD EFGH** in Figure 7 (a), three tetrahedrons **DBC G**, **DGCH**, **DGHI**, and a hexahedron **ABDEFGIJ** are created as shown in Figure 7 (b).

The shear transformation splits two quadrilateral faces of a hexahedron into triangles by adding a diagonal edge. In Figure 7 (a), quadrilaterals **HIDC** and **CBGH** are split. There are two possible tessellations for each quadrilateral. However, when such a quadrilateral is a non-conforming quadrilateral, only one of the two tessellations can conform two tetrahedrons connected to the quadrilateral. In such a case, an appropriate tessellation must be chosen, and three new tetrahedrons must conform to such a tessellation.

One shear transformation may yield subsequent shear transformations when a sheared quadrilateral face of the hexahedron is connected to another hexahedron. For example, if the shear transformation is applied to hexahedron **DABCIFGH** of Figure 8 (a) and transformed into hexahedron **ABDEFGIJ** as shown in Figure 8 (b), quadrilaterals **JIGF** and **IHGF** violate the relaxed conformity criteria because the two quadrilaterals share only three nodes. To resolve this violation, the shear transformation must be applied to the adjacent hexahedron connected to quadrilateral **IHGF** as shown in Figure 8 (c). Furthermore, the second shear transformation may yield the third shear transformation, and the series of transformations continues until all violations are resolved, or the series of transformations must be cancelled if a violation cannot finally be resolved.

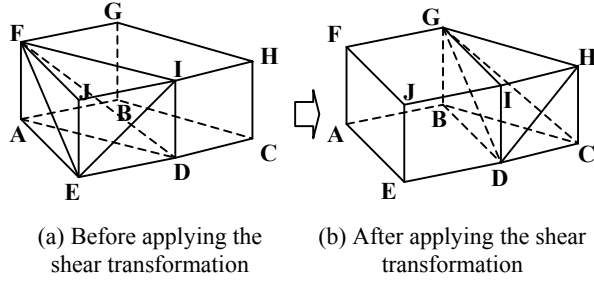


(a)  $n$  must be two in this case.



(b)  $n$  must be one in this case.

**Figure 6 In a PQ-Rn transformation,  $n$  is constrained by the tetrahedrons connected to the quadrilateral face using node R and not using edge PQ**

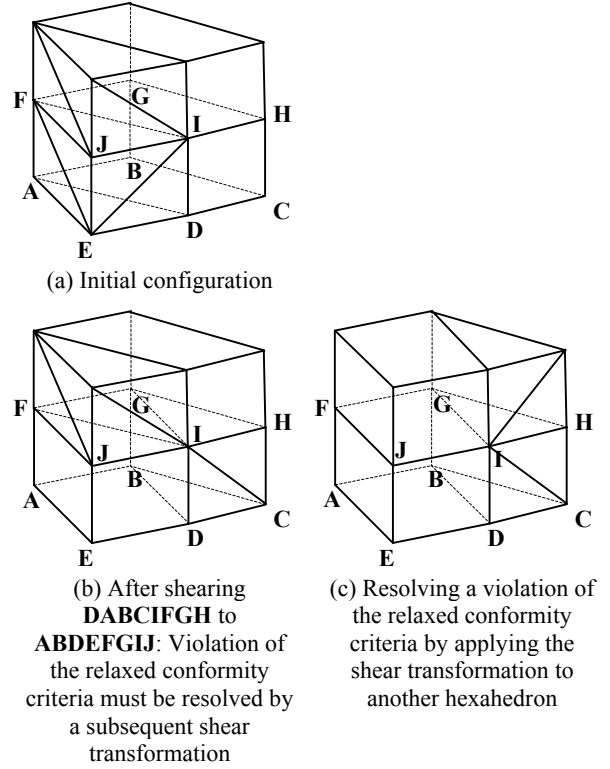


**Figure 7 The shear transformation: shearing hexahedron ABCDIFGH to ABDEJFGI**

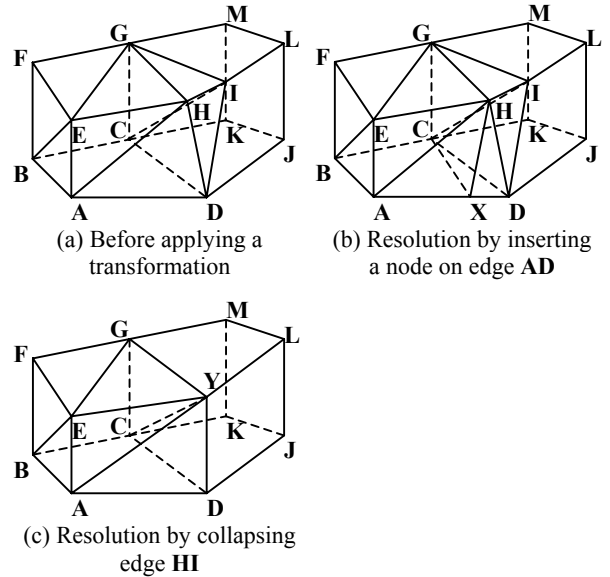
## 5. INCREASING HEXAHEDRONS VIA TOPOLOGICAL TRANSFORMATIONS

This section explains three typical patterns in which the number and the volume of hexahedrons are increased by applying the topological transformations explained in Section 4. The pattern shown in Figure 9 (a) can be solved either by collapsing edge **HI** or inserting a node on edge **AD**. Figure 9 (a) shows a portion of a hex-dominant mesh consisting of a hexahedron on the right and tetrahedrons filling the remainder of the portion. If tetrahedrons left of quadrilateral **CDHG** in Figure 9 (a), are converted to hexahedron **ABCDEFGH**, quadrilaterals **CDHG** and **CDIG** share only three nodes violating the relaxed conformity criteria. If node **X** is inserted on edge **AD** as shown in Figure 9 (b), tetrahedrons left of quadrilateral **XCGH** can be converted to hexahedron **ABCXEFGH**, and tetrahedrons filling between hexahedrons **ABCXEFGH** and **DCKJIGML** can be converted to a prism. Another resolution for this case is to collapse edge **HI**. If edge **HI** is collapsed into node **Y** as shown in Figure 9 (c), tetrahedrons left of **DCGY** can be converted to hexahedron **ABCDEFGY**.

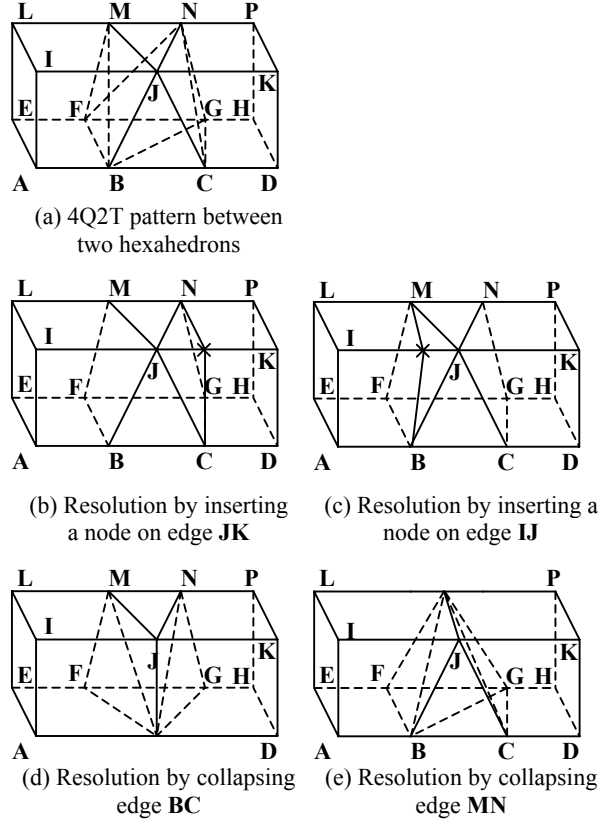
Figure 10 (a) shows a pattern in which some tetrahedrons are trapped between two hexahedrons on the left and right. An extra node is needed to convert the tetrahedrons to a hexahedron, or a node must be deleted to convert tetrahedrons to a prism. Since the volume between the two hexahedrons is enclosed by four quadrilaterals and two triangles, we call it a *4Q2T pattern*. The 4Q2T pattern shown in Figure 10 (a) can be resolved by inserting a node on either edge **JK** as shown in Figure 10 (b) or **IJ** as shown in Figure 10 (c). If a node is inserted on edge **JK** or **IJ**, tetrahedrons between the two hexahedrons will be converted to a hexahedron. Another resolution is achieved by collapsing edge **BC** as shown in Figure 10 (d) or **MN** as shown in Figure 10 (e). If edge **BC** or **MN** is collapsed, tetrahedrons between the two hexahedrons will be converted to a prism.



**Figure 8 One shear transformation may yield subsequent shear transformations (not all internal edges are drawn).**



**Figure 9 A case that can be solved either by inserting a node on edge AD or collapsing edge HI (not all internal edges are drawn).**

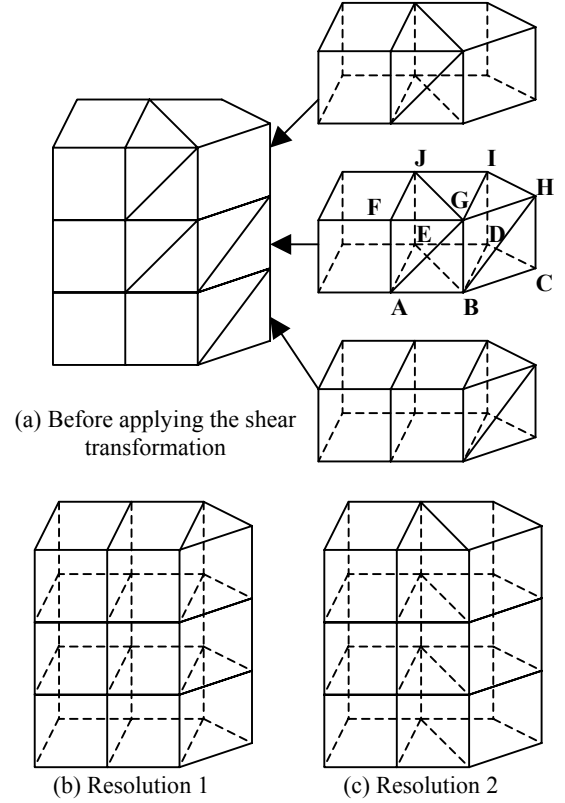


**Figure 10 Resolving a 4Q2T pattern (not all internal edges are drawn)**

Figure 11 shows a pattern that can be resolved by the shear transformation. Figure 11 shows three rows of elements, the top row has two hexahedrons on the left and right, and the mid row has a hexahedron on the left, and the bottom row has two hexahedrons on the left, and the remaining portion is filled with tetrahedrons. If tetrahedrons left of **GIDB** in the mid row are converted to hexahedron **AEDBFJIG**, quadrilateral **FJIG** shares only three nodes with the bottom face of the right hexahedron in the top row, and if tetrahedrons right of **GJEB** in the middle row are converted to hexahedron **BEDCGJIH**, quadrilateral **BEDC** shares only three nodes with the top quadrilateral of the right hexahedron in the bottom row; either case creates a violation of the relaxed conformity criteria.

If the right hexahedron in the top row is sheared so that the top row pattern matches the bottom row, tetrahedrons left of **GIDB** in the middle row can be converted to hexahedron **AEDBFJIG**, and the remaining tetrahedrons will be converted to prisms as shown in Figure 11 (b). Or if the right hexahedron in the bottom row is sheared so that the bottom row pattern matches the top row, tetrahedrons right of **GJEB** in the middle row can be converted to hexahedron **BEDCGJIH** and remaining tetrahedrons will be converted to prisms as shown in Figure 11 (c).

A pattern sometimes appears as a combination of the above basic patterns. A sequence of transformations can often resolve such patterns.



**Figure 11 A case that can be solved by the shear transformation (not all edges are drawn)**

## 6. RESULTS

Figure 12, Figure 13, and Figure 14 show experimental results of the method. For each example, an input hex-dominant mesh is created by the method presented in [10]. Each example includes screenshots of hex-dominant meshes before and after applying the transformations, remaining tetrahedrons before and after applying the transformations, statistics before and after applying the transformations.

In these example cases, the percentage of the number and volume of hexahedrons increases, and the number and volume of tetrahedrons is reduced by the transformations. The percentage of the number of hexahedrons increases from 30% to 50% in the hex-dominant mesh shown in Figure 12, from 32% to 46% in the hex-dominant mesh shown in Figure 13, from 25% to 34% in the hex-dominant mesh shown in Figure 14, and from 33% to 43% in the hex-dominant mesh shown in Figure 15. The percentage of the volume of hexahedrons increases from 67% to 82% in the hex-dominant mesh shown in Figure 12, from 70% to 80% in the hex-dominant mesh shown in Figure 13, from 66% to 76% in the hex-dominant mesh shown in Figure 14, and from 71% to 80% in the hex-dominant mesh shown in Figure 15.

These results show that the method effectively increases the number and volume of hexahedrons, and decreases the number and volume of tetrahedrons.

## 7. DISCUSSIONS

The three topological transformations described in Section 4 deal only with hexahedrons and tetrahedrons. Hence, when the input hex-dominant mesh includes prisms and pyramids, all of them must first be subdivided into tetrahedrons. Or, the three transformations cannot be performed near a prism or a pyramid.

The transformations deal only with hexahedrons and tetrahedrons because there are too many configurations of neighboring elements if pyramids and prisms are included. There are many variations of a configuration of neighboring elements even with only hexahedrons and tetrahedrons. The transformations would become too complex due to an overwhelmingly large number of variations of neighboring element configuration, if prisms and pyramids are considered. Thus it was deemed reasonable to limit transformations for hexahedrons and tetrahedrons, and subdivide prisms and pyramids in the input mesh into tetrahedrons first, and re-create them after the transformations are performed.

The result of the proposed method depends on the order of the transformations. Some transformations performed earlier may interfere with further transformations, and fewer hexahedrons can be created as a result of this interference. In the current implementation, the program searches and resolves the patterns shown in Figure 9 and Figure 11 first. It then searches and resolves the patterns shown in Figure 10. It iterates the search and resolution until no such pattern is found. However, it is very difficult to find the optimal order of the transformations, and thus it becomes a subject for future research.

The effectiveness of the method also depends on an input hex-dominant mesh. However, we could only test the method with a hex-dominant mesh created by the method presented in [10]. The proposed method works well when there are some islands of tetrahedrons, which gives some freedom for the transformations. Therefore, we expect that the proposed method will also work well for a hex-dominant mesh created by Meshkat and Talmor's method [15]. However, the proposed method may not perform well for advancing front type methods [11, 12, 14] because such methods tend to create less islands of tetrahedrons and the remaining tetrahedrons tend to be flatter. The relation between the effectiveness of the proposed method and hex-dominant mesh generation algorithms is another issue for future research.

Also, when there is more than one option for resolving a case, the program must decide which option to choose. And, one choice may interfere with further transformations yielding fewer hexahedrons. The current implementation chooses the option that gives better quality elements measured by the scaled Jacobian [18, 19]. However, choosing a less but acceptable quality element may yield

more hexahedrons. Choosing an option to resolve a case also becomes an issue for future research.

In addition to the three transformations described in Section 4, there can be more possible transformations. There may be a better strategy of applying the transformations than the strategy described in Section 5. An ultimate goal of this method is to convert the input hex-dominant mesh into a fully conformed hex-dominant mesh without using pyramids while filling most of the volume with hexahedrons. However, it is unclear if such transformations and strategies are possible, and thus it is another issue for future research.

## 8. CONCLUSIONS

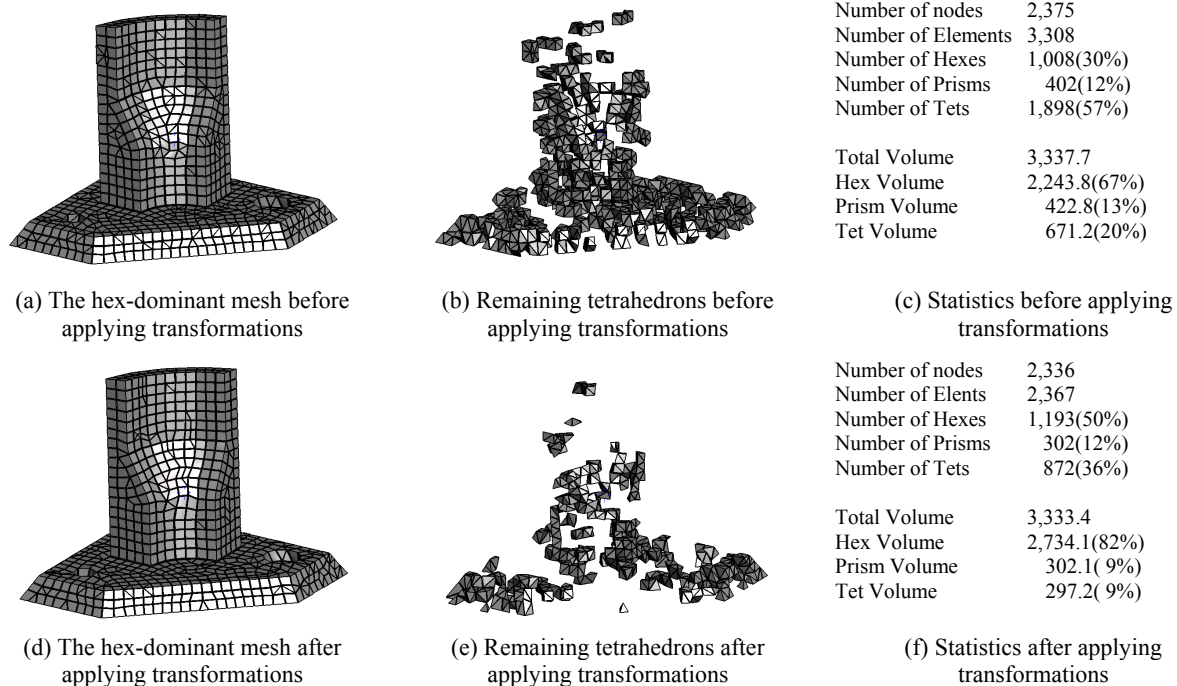
This paper has presented a new method for increasing the number and volume of hexahedrons in a hex-dominant mesh by applying topological transformations. Three types of transformations are presented, and by applying them strategically the number and volume of hexahedrons in the hex-dominant mesh increase. Some experimental results show the method works effectively, and it increases the practical value of a hex-dominant mesh. The experimental results show the number of hexahedrons increased by about 10% to 20%, yielding hex-dominant meshes with a 70 to 90% hexahedron volume ratio.

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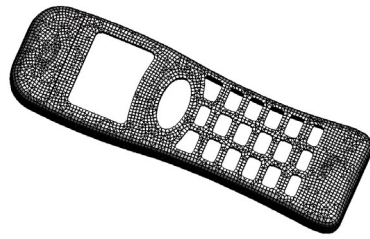
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**Figure 12 Hex-dominant mesh of a mechanical part**



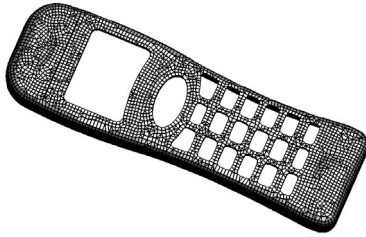
(a) The hex-dominant mesh before applying transformations



(b) Remaining tetrahedrons before applying transformations

Number of nodes	10,462
Number of Elements	10,086
Number of Hexes	3,263(32%)
Number of Prisms	1,459(14%)
Number of Tets	5,364(53%)
Total Volume	5,843.8
Hex Volume	4,067.4(70%)
Prism Volume	846.0(14%)
Tet Volume	930.4(16%)

(c) Statistics before applying transformations



(d) The hex-dominant mesh after applying transformations

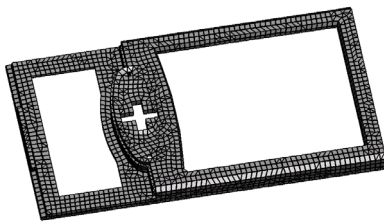


(e) Remaining tetrahedrons after applying transformations

Number of nodes	10,221
Number of Elements	7,979
Number of Hexes	3,684 ( 46%)
Number of Prisms	1,028 ( 12%)
Number of Tets	3,267 ( 40%)
Total Volume	5,818.5
Hex Volume	4,682.8(80%)
Prism Volume	596.8(10%)
Tet Volume	538.9( 9%)

(f) Statistics after applying transformations

**Figure 13 Hex-dominant mesh of a cell-phone case**



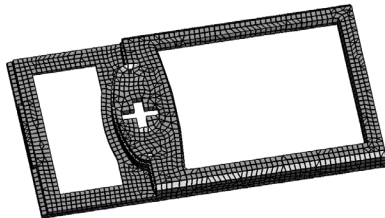
(a) The hex-dominant mesh before applying transformations



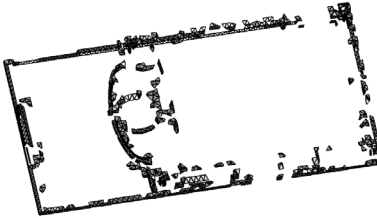
(b) Remaining tetrahedrons before applying transformations

Number of nodes	4,918
Number of Elements	5,382
Number of Hexes	1,374(25%)
Number of Prisms	695(12%)
Number of Tets	3,313(61%)
Total Volume	12,263.8
Hex Volume	8,117.9(66%)
Prism Volume	1,821.2(15%)
Tet Volume	2,324.8(19%)

(c) Statistics before applying transformations



(d) The hex-dominant mesh after applying transformations

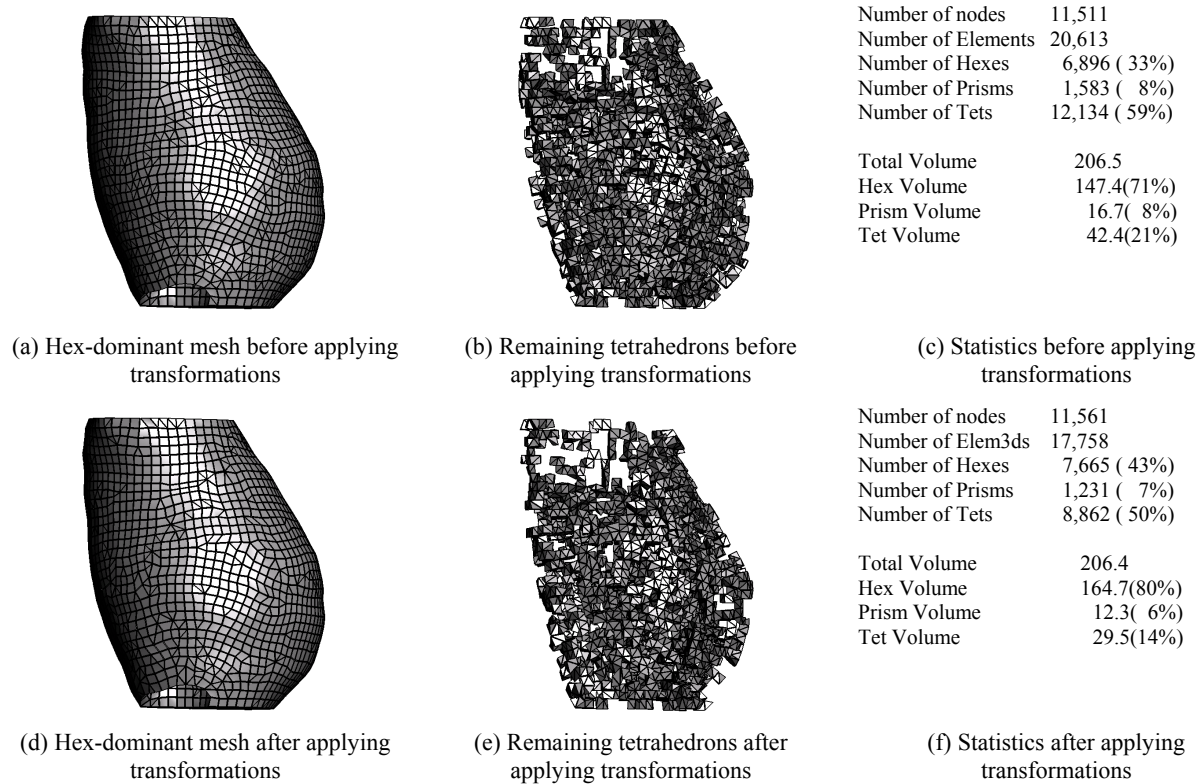


(e) Remaining tetrahedrons after applying transformations

Number of nodes	4,849
Number of Elements	4,530
Number of Hexes	1,576(34%)
Number of Prisms	502(11%)
Number of Tets	2,452(54%)
Total Volume	12,240.5
Hex Volume	9,363.1(76%)
Prism Volume	1,316.4(11%)
Tet Volume	1,561.0(13%)

(f) Statistics after applying transformations

**Figure 14 Hex-dominant mesh of a PDA case**



**Figure 15 Hex-dominant mesh of a blood vessel for analysis of aneurysm: Due to varying cross-section, it is difficult to create a mesh of this geometry with a conventional method**