

# Automatic All-Hex Mesh Generation of Thin-Walled Solids via a Conformal Pyramid-less Hex, Prism, and Tet Mixed Mesh \*

Soji Yamakawa<sup>1</sup> and Kenji Shimada<sup>2</sup>  
The Department of Mechanical Engineering  
Carnegie Mellon University

<sup>1</sup>soji@andrew.cmu.edu

<sup>2</sup>shimada@cmu.edu

## Abstract

A new computational method for creating all-hex meshes of thin-walled solids is presented. The proposed method creates an all-hex mesh in a three-step process. First, a tet mesh of the target geometric domain is created, and a layer of prism elements is inserted on the boundary of the mesh. Second, a sequence of novel topological transformations and smoothing operations is applied to reduce the number of tet elements and increase the number of hex elements. The topological transformations maintain conformity of the hex, prism, and tet mixed mesh. The proposed topological transformations do not introduce pyramid elements, and therefore avoids problems imposed by pyramid elements. Third and finally, the mixed mesh is subdivided into an all-hex mesh by applying mid-point subdivision templates. Experimental results show that the proposed method creates high-quality all-hex meshes.

## 1 Introduction

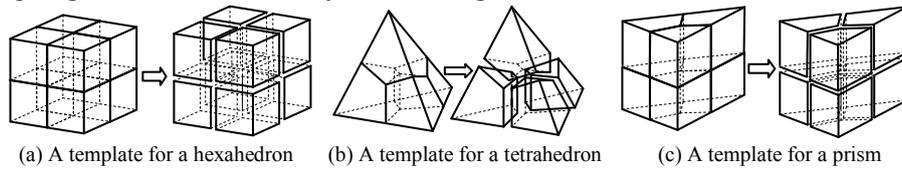
This paper describes a new computational method for creating high-quality all-hex meshes of thin-walled solids. The proposed method creates an all-hex mesh in three steps: (1) prism-tet pillowing, (2) conformal transformation, and (3) all-hex mesh subdivision. In Step (1), the proposed method creates an all-tet mesh of the input geometric domain, and a layer of prism elements is inserted on the boundary of the tet mesh. In Step (2), a sequence of topological transformations and smoothing operations is applied so that the number of tet elements between prism elements is reduced, and some prism elements are converted to hex elements. In Step (3), the elements of the mixed mesh are

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(\*) Provisional patent (61/520,795) has been filed on the technique described in this paper.

subdivided into all-hex elements by applying mid-point subdivision templates as shown in Figure 1.

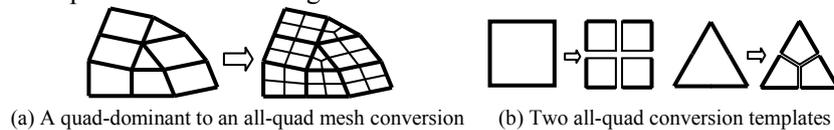
The main contribution of this research is in the second step. Conformal transformations reduce the number of tet elements and increase the number of hex elements while maintaining mesh conformity. There have been no previously-published topological transformations that transform a hex, prism, and tet mixed mesh without inserting pyramid elements. Pyramid elements are problematic in two ways: (1) many finite-element solvers reject pyramid elements and thus the mixed mesh itself becomes unusable for such solvers, and (2) unlike other three types of volumetric elements, hex, prism, and tet, a pyramid element cannot be subdivided into all-hex elements with a simple mid-point subdivision template (see Figure 1). Applying a pyramid subdivision template [1] substantially increases element count and decreases mesh quality. Problems imposed by pyramid elements are avoided in the proposed new topological transformations by not inserting such elements.



**Figure 1 Mid-point subdivision templates**

All-hex meshes tend to yield more accurate finite-element solutions than all-tet meshes, however, automatically generating all-hex meshes is much more challenging than generating all-tet meshes. The latter can be performed in a straight forward manner [2-4]. Despite two decades of research, an automated all-hex mesh generator - so-called ‘push-button’ all-hex mesh generator, has not yet been developed

In the effort to develop an all-hex mesh-generation scheme, mesh-conversion approach, which first creates a hex, prism, and tet mixed mesh and then converts it into an all-hex mesh, attracted much early attention since a similar approach worked well in 2D. In 2D meshing, a quad-triangle mixed mesh can be subdivided into an all-quad mesh by applying mid-point subdivision templates as shown in Figure 2.



**Figure 2 2D mixed mesh to all-quadrilateral mesh conversion**

Due to the algorithm's similarity, numerous attempts have been made to translate the 2D method into 3D. However, conversion from a 3D mixed mesh to an all-hex mesh poses additional, a higher level of challenges. In 2D, all

elements interface over the same type of boundary, an edge, whereas in 3D, elements can interface over either a triangular or a quadrilateral face. Since a tet element consists of triangular faces alone, and a hex element of quad faces alone, it is impossible to make a conformal connection between these two types of elements.

Although numerous potential solutions have been proposed [1, 5-9], none of them can solve the tet-hex connection problem without using a complex element-subdivision template, which substantially increases the element count and creates low-quality elements.

The proposed method, to the authors' knowledge, is the first all-hex mesh-conversion approach that does not require a complex subdivision template. By avoiding a complex subdivision template, the proposed method can create a high-quality all-hex mesh with no areas of unintended non-uniform node distribution.

## 2 Overview of the Proposed Method

To assist the explanation and comprehension of the proposed method, the 2D analogue is presented. The 2D analogue of the proposed method creates an all-quad mesh in three steps:

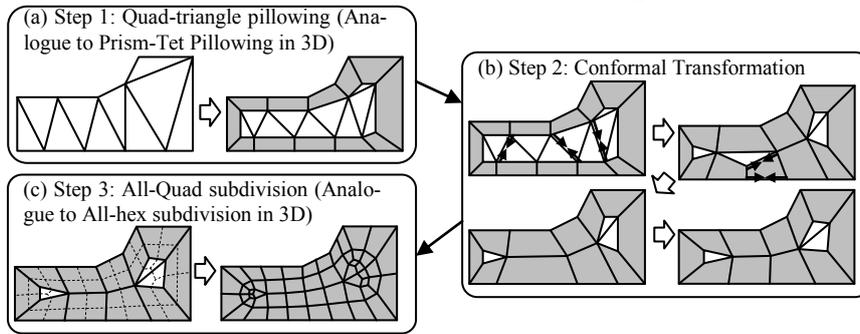
- Step 1: Quad-triangle pillowing (prism-tet pillowing in 3D)
- Step 2: Conformal transformation
- Step 3: All-quad subdivision (all-hex subdivision in 3D)

In Step 1, an all-triangular mesh (tet mesh in 3D) of the input geometric domain is created, and a layer of quad elements (prism elements in 3D) is inserted on the boundary of the triangular mesh as shown in Figure 3 (a). In Step 2, a sequence of topological transformation, detailed in Section 4, and smoothing operations are applied as shown in Figure 3 (b). One of the most important goals of this step is to reduce the number of triangular (tet in 3D) elements. In 3D, some prism elements are also converted to hex elements. In Step 3, the elements included in the mesh are subdivided into all-quad (all-hex in 3D) elements as shown in Figure 3 (c).

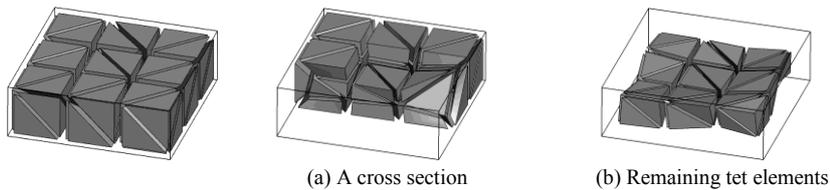
Step 1 of the 3D solution generates a tet mesh of the input geometric domain as shown in Figure 4 and inserts a layer of prism elements on the boundary of the tet mesh as shown in Figure 5.

Step 2 applies a sequence of topological transformations and smoothing operations resulting in a reduction of the number of tet elements and an increase in the number of hex elements while maintaining the mesh conformity. A variety of topological transformations are applied in order to achieve the two goals. For example, edge-collapse transformations eliminate tet elements lying between prism elements as shown in Figure 6. These transformations in addi-

tion to other topological transformations and smoothing operations are applied until no more tet element can be removed as shown in Figure 7.

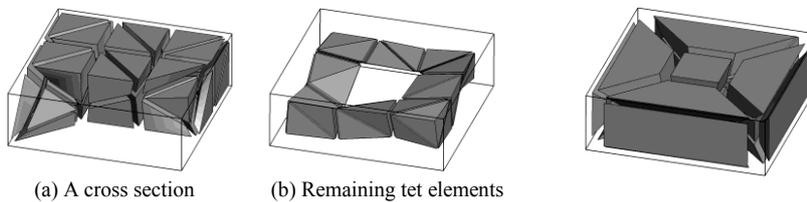


**Figure 3 2D analogue of the proposed method**



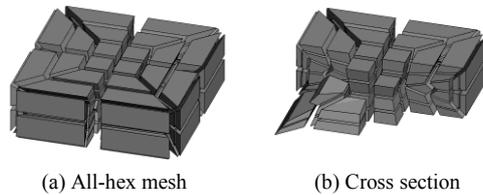
**Figure 4 Initial tet mesh**

**Figure 5 Inserting a layer of prism elements**



**Figure 6 After some tet elements are collapsed**

**Figure 7 After applying sequence of smoothing and other topological transformations**



**Figure 8 All-hex mesh converted from a conformal pyramid-less hex, tet, and prism mixed mesh**

Step 3 produces an all-hex mesh by subdividing the mixed elements into all-hex elements by applying mid-point subdivision templates as shown in

Figure 8. The mesh remains conformal through each step of the algorithm and no problematic pyramid element is introduced.

Step 1 can be done by a conventional tet mesh-generation scheme [2, 3, 10, 11] and a conventional boundary-layer element-generation scheme [12, 13]. Unlike the boundary-layer element generation for a CFD application, only one layer of prism elements is created in this step. For smoothing operations in Step 2, the proposed method uses combination of schemes including linear-programming method [14, 15], volume equalization [9], and angle-based smoothing method [16]. Mid-point subdivision templates used in Step 3 has long been known to exist.

The uniqueness and power of the proposed method is in the novel topological transformations used in Step 2. These transformations reduce the number of tet elements and increase the number of hex elements while maintaining mesh conformity. No existing topological transformation methods had been able to transform a hex, prism, and tet mixed mesh without introducing pyramid elements. This problem was substantially limiting the usability of a mixed meshes and made all-hex conversion very difficult. Since the proposed topological transformation schemes do not introduce pyramid elements, it effectively avoids problems imposed by the pyramid elements. Details of the topological transformations are explained in Section 4.

### 3 Prism Chain and Prism-Chain Pair

Many of the topological transformations used by the proposed method transform a collection of prism elements called a *prism-chain pair*. A prism-chain pair is a pair of sets of prism elements called *prism chains*. A prism chain is defined as a single prism element or a set of prism elements that are connected to at least one other member of the set by a triangular face.

Two prism chains may form a prism-chain pair when the following conditions are met: (1) the prism chains consist of same number of elements, (2) each of the elements in one prism chain has an element in the other prism chain that shares a quadrilateral face, and (3) each of the triangular faces of one prism chain has a triangular face in the other prism chain that shares an edge. A prism-chain pair has a column of quadrilateral faces, called *gluing faces*, each of which is shared by a prism element in one prism chain and a prism element in the other prism chain. A prism-chain pair also has a column of edges, each of which is shared by two triangular faces within the prism-chain pair. Such edges are called *rib edges*. Figure 9 shows an example of a prism-chain pair with two prism chains each of which consist of three prism elements.

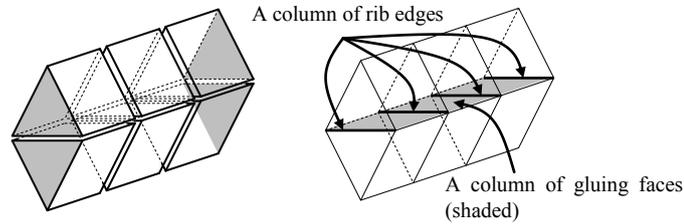


Figure 9 A prism-chain pair with two prism chains that consist of three prisms each.

## 4 Topological Transformations

This section explains in detail the topological transformations used for reducing the number of tet elements and increasing the number of prism and hex elements. The proposed method uses following eight types of transformations:

- Edge Collapse
- Prism Swap
- Prism Collapse
- Prism Split
- Clamp
- Diamond Collapse
- Half- and Full-Wedge Collapse
- Prism Paring

Mesh conformity is guaranteed through the topological transformations.

### Edge Collapse

An edge used only by tet elements can be collapsed, or all tet elements using the edge are deleted and two nodes of the edge are merged to one. For example, edge **AB** in Figure 10 (a) is shared by six tet elements and can be collapsed. The transformation connects the prism elements sharing node **A** and the prism elements sharing node **B** as shown in Figure 10 (b).

The main purpose of this transformation is to remove tet elements lying between prism elements. If multiple tet elements exist between opposing prism elements, the transformation needs to be applied multiple times to connect two opposing prism elements.

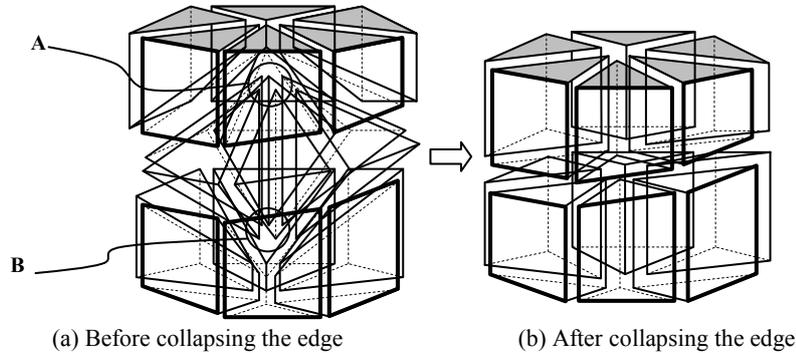


Figure 10 Example of edge-collapse transformation

### Prism Swap

The prism-swap transformation re-connects, or swaps, a column of gluing faces in a prism-chain pair. The two prism elements sharing a gluing face have a combined four edges that are not used by any of the four triangular faces of the two prism elements. Two of the four edges are connected by a gluing face, and the remaining two are not directly connected. The prism-swap transformation replaces the two prism elements with two new prism elements such that new gluing face connects the two edges that were not connected in the original configuration and disconnects the edges that were originally connected as shown in Figure 11. Shaded faces in the figure are exterior faces and are not connected to other elements.

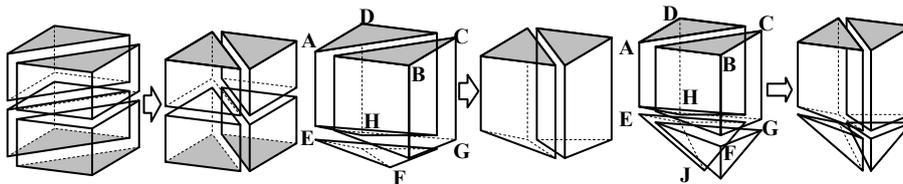


Figure 11 Prism-swap transformation

Figure 12 A variation of prism-swap transformation

Figure 13 Another variation of prism-swap transformation

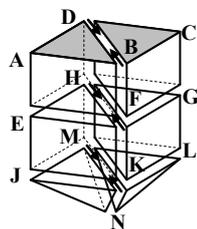
If a tet element shares four nodes of one end of a prism-chain pair, the tet element is removed by this transformation. In Figure 12, a sub-volume of the mesh is initially filled with a pair of prism elements **ABCEFG** and **ACDEGH** (**ABC** and **ACD** are exterior faces) and a tet element **FHGE**. The three elements can be replaced with two prism elements **ABDEFH** and **BCDFGH**.

When two tet elements sharing a triangular face are connected to one end of a prism-chain pair, the two tet elements are also replaced with two new tet elements by the prism-swap transformation such that the conformity is maintained. In Figure 13, a sub-volume of a mesh is initially filled with two prism elements **ABCEFG** and **ACDEGH** and two tet elements **FEJG** and **EHJG**. The two prism elements are replaced with two new prism elements **ABDEFH** and **BCDFGH**, and the two tet elements are replaced with two new tet elements **FEJH** and **FHJG**.

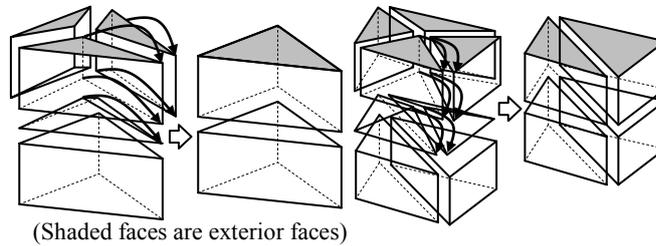
### Prism Collapse

The prism-collapse transformation collapses each rib edge in the prism-chain pair into one node, and essentially deletes all the elements sharing a rib edge. A rib edge at an end of the prism-chain pair can be used by tet elements, which are deleted by the prism collapse transformation. If a rib edge is used by a non-tet element that is not included in the prism-chain pair, this transformation cannot be applied.

Prism and tet elements shown in Figure 14 can be collapsed using the prism collapse transform. Prism elements **ABDEFH**, **BCDFGH**, **EFHJKM**, **FGHKLM**, and tet elements **JKMN**, **KLMN**, can be deleted by collapsing edges **BD**, **FH**, and **KM** into one node each.



**Figure 14** A prism-chain pair and connected tet elements that can be collapsed



**Figure 15** An application of the prism-collapse transformation

**Figure 16** An application of the prism-collapse transformation

This transformation is useful when three prism elements share an edge, all three prism elements are connected to a single tet element, and the other side of each prism element is exposed to the exterior of the mesh. Figure 15 shows this configuration and illustrates how the three prism elements are collapsed into one and the tet element is deleted.

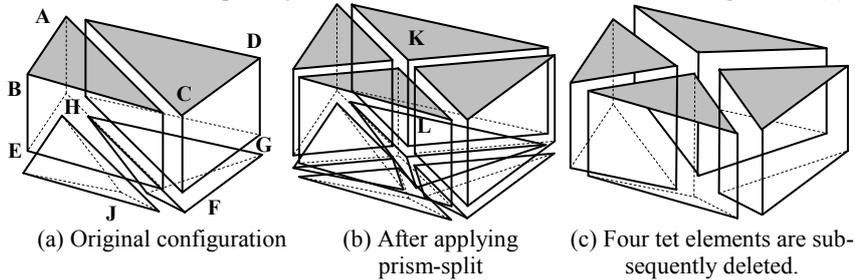
This transformation is also useful when four prism elements share an edge, two of them are connected to a single tet element, and the remaining two are connected to another single tet element. This configuration, shown in Figure

16, can be collapsed from four prism and two tet elements to two prism elements.

### Prism Split

The prism-split transformation splits each prism element in a prism-chain pair into two prism elements by inserting a node on each rib edge. If tet elements are using a rib edge at an end of the prism-chain pair, each of those tet elements are also split into two.

For example, consider two prism elements **ABCEF**H and **ACDHFG** that share a gluing face **AHFC** as shown in Figure 17 (a) and two tet elements **EFHJ** and **HFGJ** that are connected to the pair of the two prism elements. Node **K** is inserted on edge **AC** and node **L** is inserted on **HF**, prism element **ABCEF**H is split into two prism elements **ABKHEL** and **BCKEFL**, prism element **ACDHFG** is split into two prism elements **AKDHLG** and **CDKFGL**, tetrahedron **EFHJ** is split into two tet elements **ELHJ** and **EFLJ**, and tetrahedron **HFGJ** is split into two tet elements **HLGJ** and **FGLJ** as shown in Figure 17 (b). Node **L** can be then merged to node **J** and the four tet elements can subsequently be deleted in this case as shown in Figure 17 (c).



**Figure 17 The prism-split transformation**

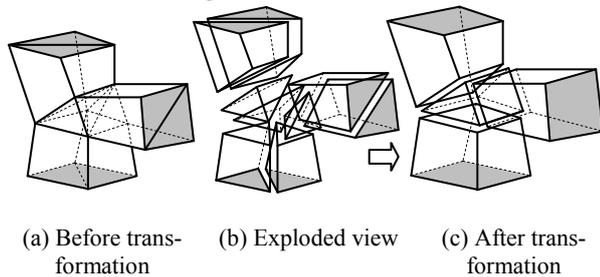
### Clamp

The clamp transformation converts three tet elements with three connected prism-chain pairs into one prism element with three chains of hex elements. This transformation reduces number of tet elements and converts pairs of prism elements into single hex elements.

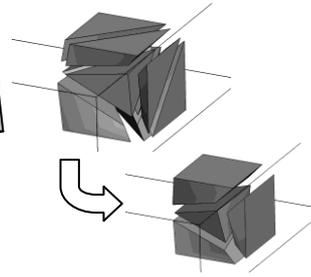
The following four conditions are needed to perform the clamp transformation: (1) three tet elements share an edge and are comprised of six nodes, (2) three prism-chain pairs are connected to the three tet elements, (3) the other side of the three prism chain pairs are exposed to the exterior of the mesh, and (4) two triangular faces of the three tet element cluster that are exposed out-

side the three tet elements and not connected to the three prism chain pairs do not share any nodes.

Figure 18 (a) shows a configuration that satisfies the above conditions which is converted into one prism and three hex elements after using the clamp transformation. Figure 19 shows a typical pattern on which the transformation can be applied. This pattern often appears on models near the corners of thin plate features. The clamp transformation reduces the number of tet element in such geometric constructs.



**Figure 18** An example of the clamp transformation

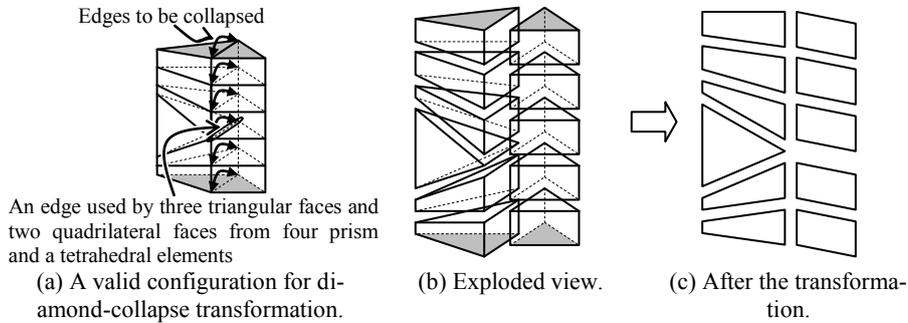


**Figure 19** A typical configuration that the clamp transformation can be applied

### Diamond Collapse

The diamond-collapse transformation collapses and deletes three prism chains and a tet element. The transformation can be applied when an edge,  $E_0$ , is shared by three triangular and two quadrilateral faces such that the two quadrilateral faces are separated by one of the triangular faces and that the 3D elements that share edge  $E_0$  are four prism elements plus a tet element, as shown in Figure 20 (a). Two of the three prism chains to be deleted by this transformation start from the triangular faces of the tet element sharing  $E_0$  and continue to the boundary of the mesh. The third prism chain includes the third triangular face sharing  $E_0$ . Each prism element in the third prism chain shares a quadrilateral with a prism element in one of the first two chains. Both ends of the third prism chain must be exposed to the exterior of the boundary. If the above conditions are satisfied, edge  $E_0$  and each edge shared by two triangular faces of the three prism chains can be collapsed to a node. The diamond-collapse transformation maintains mesh conformity.

Figure 20 illustrates an instance of the diamond-collapse transformation applied to a set of prism and tet elements resulting in only quadrilateral and triangular faces after transformation.

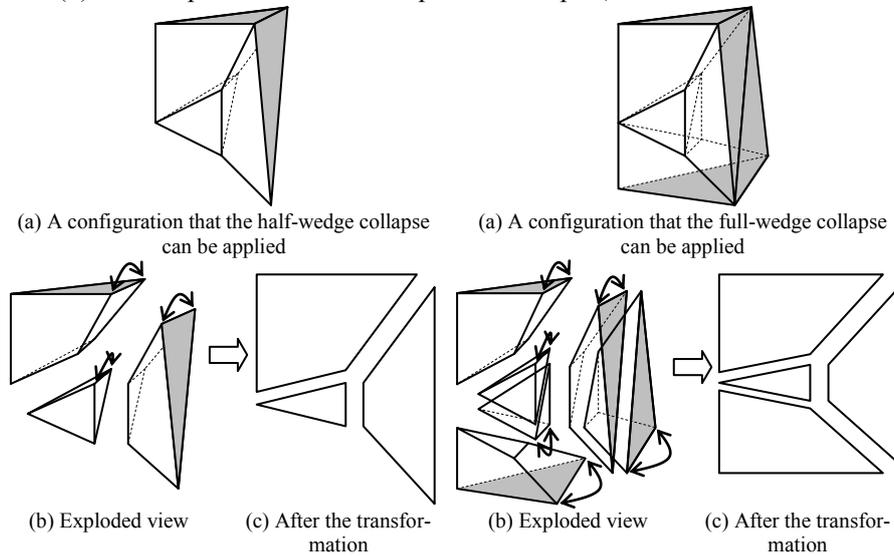


**Figure 20 Diamond collapse (Shaded faces are exposed to the exterior of the mesh.)**

### Half- and Full- Wedge Collapse

The half-wedge-collapse transformation collapses and deletes two prism chains and a tet element. When the following three conditions are met:

- (1) one edge is shared by two triangular faces and a quadrilateral face from two prism and one tet elements,
- (2) the other end of the two prism chains are exposed to the mesh boundary, and
- (3) the two prism chains form a prism-chain pair,



**Figure 21 Half-wedge collapse transformation (Shaded faces are exterior faces.)**

**Figure 22 Full-wedge collapse transformation**

Each of the edges in the prism-chain pair shared by two triangular faces is collapsed into one node, and the tet element and the prism elements in the prism chains are deleted as shown in Figure 21.

Figure 22 illustrates an instance of a commonly appearing variant configuration that can be addressed with the half-wedge collapse transformation. The configuration is comprised of two separate valid half-wedge collapse transformation candidates that can be optimized in a single operation such that each of the edges that would be collapsed separately can be collapsed at the same time.

### Prism Paring

Prism-pairing transformation merges two prisms sharing a quadrilateral face included in a prism-chain pair as shown in Figure 23. This transformation is applicable if one of the following two conditions is satisfied: (1) both end faces of the prism-chain pair are exposed to the exterior of the mesh, or (2) the prism-chain pair is a closed prism-chain pair, i.e., none of the triangular faces is exposed to the outside of the prism-chain pair. If neither condition is satisfied this transformation will result in a non conformal mesh, and cannot be applied.

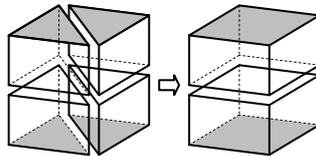


Figure 23 Prism-pairing transformation

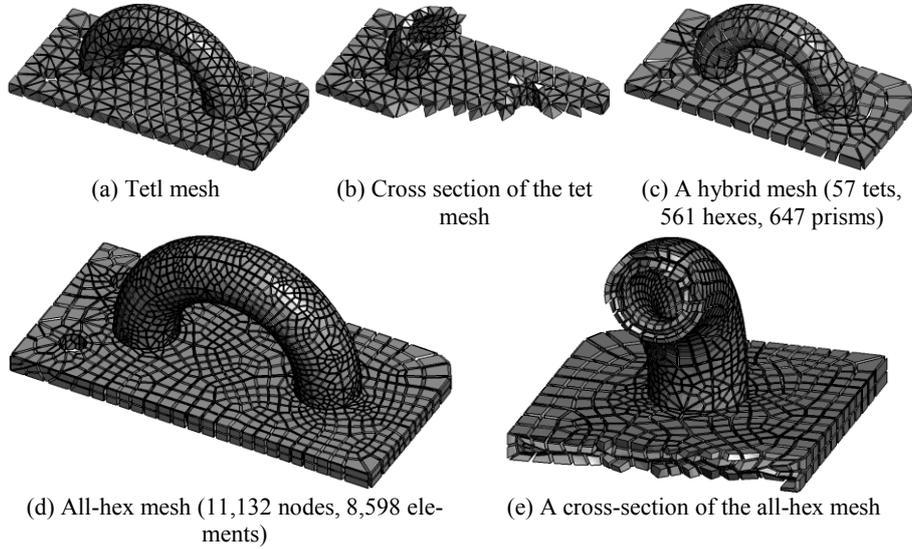
### Maintaining Element Quality and Element Size

It is not trivial to estimate the resulting element quality and element size before actually applying the above-mentioned transformations. The program, therefore, checks affected elements after each transformation, and if an element violates the user-specified quality criteria, the transformation is retracted.

When a given geometric domain consists of non-thin-walled, or thick, parts, it becomes difficult to remove many of the tet elements lying between opposing prism elements without violating quality criteria. Tet elements remain un-removed in such thick parts, and hex elements subdivided from those remaining tet elements will have relatively lower quality.

## 5 Examples

One major advantage of the proposed method is requiring no manual volume decomposition to obtain high-quality meshes. The tet mesh of a U-shaped pipe connected to the thin plate as shown in Figure 24 demonstrates such an example. Using conventional methods, such a model must be manually decomposed into sweepable sub-domains to create a high-quality all-hex mesh. The proposed method does not require such manual decomposition. Table 1 presents metrics of the automatically generated all-hex mesh, showing that the results are of good quality.



**Figure 24 An example of non-sweepable geometry**

**Table 1 Statistics of the all-hex mesh of the non-sweepable geometry**

Scaled Jacobian and number of elements		Quad Warpage (degree) and number of quads	
1.0 to 0.9	99	0 to 10 degree	18,277
0.9 to 0.8	575	10 to 20 degree	9,182
0.8 to 0.7	1127	20 to 30 degree	367
0.7 to 0.6	1502	30 to 40 degree	5
0.6 to 0.5	1459	Greater than 40 degree	0
0.5 to 0.4	1447		
0.4 to 0.3	1302		
0.3 to 0.2	1087		
Less than 0.2	0		
<i>Minimum Scaled Jacobian: 0.20</i>		<i>Maximum Warpage Angle: 34.2 degree</i>	

The proposed method can be applied to biological models such as that of an aorta artery as shown in Figure 25 (a). The hybrid mesh shown in Figure 25 (b) is generated from the tet mesh shown in Figure 25 (a). Subdividing the

mesh yields the all-hex mesh shown in Figure 25 (c). Quality statistics of the all-hex mesh, shown in Table 2, indicate that the results are of high quality.

## 6 Reduced Mid-Point Subdivision

Prism elements inserted in the prism-tet pillowing step (the first step) makes a sheet of prism elements along the boundary of the tet mesh. An element included in such a sheet has a distinctive thickness direction. Each exterior face of a sheet has an opposite face, which is an interior face connected to a tet element of the mesh. Edges connecting an exterior face and the opposite face define the thickness direction. Since none of the proposed topological transformations breaks the sheet, the thickness direction remains the same through the conformal transformation step even after some prism elements in the sheet are converted to hex elements.

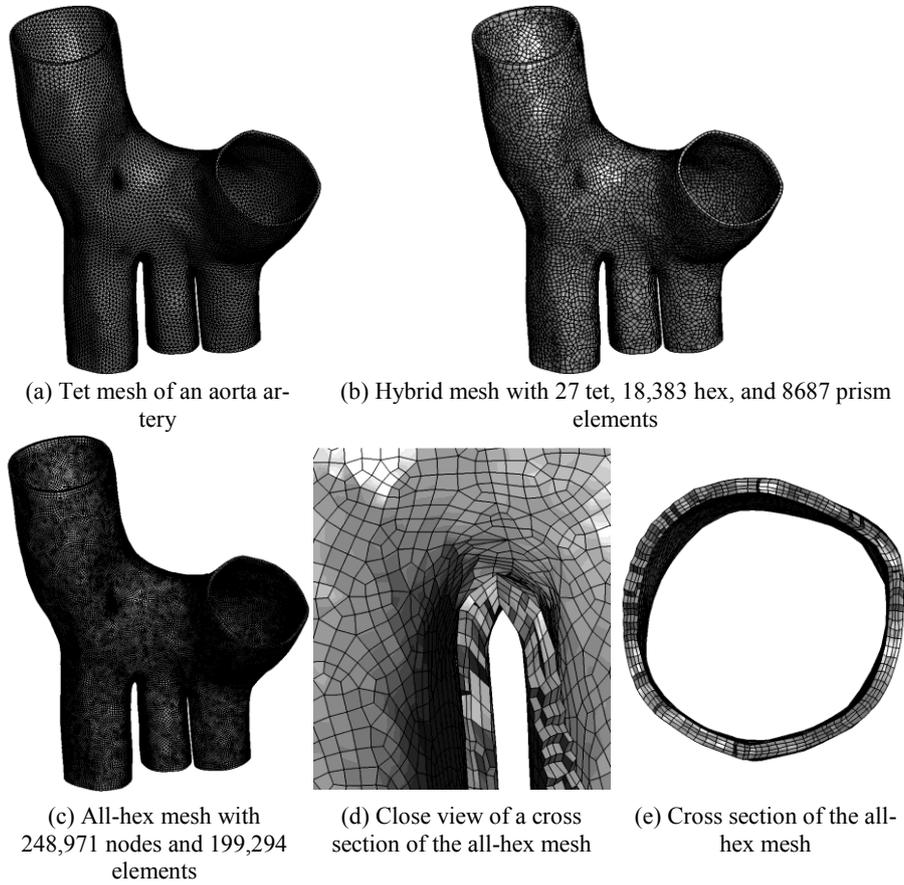
When a sheet of elements has a distinctive thickness direction, the elements do not have to be subdivided in the thickness direction in the all-hex subdivision step (the third step). For those elements, reduced mid-point subdivision templates shown in Figure 26 (a) and (b) can be applied. These subdivision templates insert a node only on the exterior face and the opposite face. No node is inserted inside the element.

The reduced mid-point subdivision templates substantially reduce the number of nodes and elements in the final all-hex mesh. However, the mesh has less freedom of the interior nodes, and the mesh-smoothing scheme may not achieve a target mesh quality due to the lack of freedom.

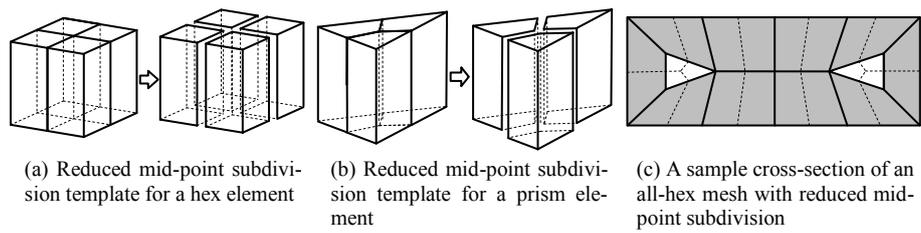
The decision of whether to use reduced mid-point subdivision should be made according to the requirements of the application in which the mesh is used.

**Table 2 Statistics of the all-hex mesh of the aorta artery**

Scaled Jacobian and number of elements		Quad Warpage (degree) and number of quads	
1.0 to 0.9	38,114	0 to 10 degree	490,215
0.9 to 0.8	83,895	10 to 20 degree	128,342
0.8 to 0.7	42,740	20 to 30 degree	22,958
0.7 to 0.6	20,042	30 to 40 degree	4558
0.6 to 0.5	7,810	40 to 50 degree	920
0.5 to 0.4	3,634	50 to 60 degree	329
0.4 to 0.3	2,286	60 to 70 degree	1
0.3 to 0.2	771	Greater than 70 degree	0
0.2 to 0.1	2		
Less than 0.1	0		
<i>Minimum Scaled Jacobian:0.199993</i>		<i>Maximum Warpage Angle:60.01 degree</i>	



**Figure 25 An aorta model**



**Figure 26 Reduced mid-point subdivision**

## 7 Discussions on the Order Dependency Issue

While the result from the above topological transformations depends on the order of transformation operations, it is difficult to find the globally optimum transformation order in a practical computational time.

Nonetheless, experiments conducted in research indicate some clues of finding a good transformation order including:

- (1) Locations where prism-collapse, prism-swap, and prism-split transformations can be applied are often isolated and do not depend on the order of the transformations.
- (2) Diamond-collapse, wedge-collapse, and clamp transformations work more effectively after tet elements are deleted as much as possible by edge-collapse, prism-collapse, prism-swap, and prism-split transformations.

From these observations, the following steps have been tested to confirm that they work well for many cases:

- 1) Apply an edge-collapse transformation when it can delete a tet element lying between two nearest nodes from opposing prism elements.
- 2) Apply prism-collapse, prism-swap, and prism-split transformations whenever the transformation can reduce the number of tet elements.
- 3) Apply smoothing.
- 4) Repeat Steps 1 to 3 until no more tet element can be deleted.
- 5) Apply clamp, diamond-collapse, and wedge-collapse transformations as many times as possible.

This strategy, a greedy-search approach targeting the maximum reduction of tet elements, gives a reasonably good result in many test cases used in this research. In some cases, however, manually tweaking the order of the transformations finds a better mesh in the end. There can be a smarter automated algorithm of finding a more ideal order of transformations.

One option is to apply greedy-search approach targeting the best mesh quality rather than the maximum reduction of tet elements. In this approach, the transformation that makes the best quality elements is prioritized in Steps 1 and 2 of the above approach. The issue to be solved to implement this approach is how to compare the element quality. In a mixed-element mesh, the element quality is measured by multiple measurements such as scaled Jacobian, quad warpage, and radius-ratio. Since those measurements are highly non-linear, simply taking a weighted sum does not give a fair evaluation. In other words, this approach is a multi-objective optimization, and further research is needed to develop the good element-comparison condition for this purpose.

Another option is a trial-and-error approach. In this approach, a greedy-search approach is applied first, from which the program searches a possible

improvements that could be made by applying topological transformations in a different order. Then the topological transformations applied in the first step are retracted and the mesh is brought back to the state immediately after the prism-tet pillowing step. Finally, the topological transformations are applied in the order that is modified based on the search. This trial-and-error loop can be repeated until no more improvement is gained. This approach suffers from the same issue as the quality-based greedy-search approach; it requires a good element-comparison condition to evaluate the improvement. Also, since this applies the conformal-transformation step multiple times, it takes longer time to generate a mesh. Further research is needed for developing a solution to these issues.

## 8 Conclusions

This paper has presented a new computational method for creating conformal pyramid-less hex, prism, and tet mixed meshes and an all-hex meshes of thin-walled solids. The proposed method creates an all-hex mesh in three steps: (1) prism-tet pillowing, (2) conformal transformation, and (3) all-hex mesh subdivision.

The significance of the new topological transformations used in Step (2) is it maintains the conformity of the hex, prism, and tet mixed mesh. Known topological transformation that modifies the topology of a mixed mesh requires pyramid elements to preserve mesh conformity. A pyramid element, however, is problematic and makes conversion from a mixed mesh to an all-hex mesh difficult. The proposed topological transformation does not introduce pyramid elements, and therefore avoids problems imposed by pyramid elements.

## References

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